

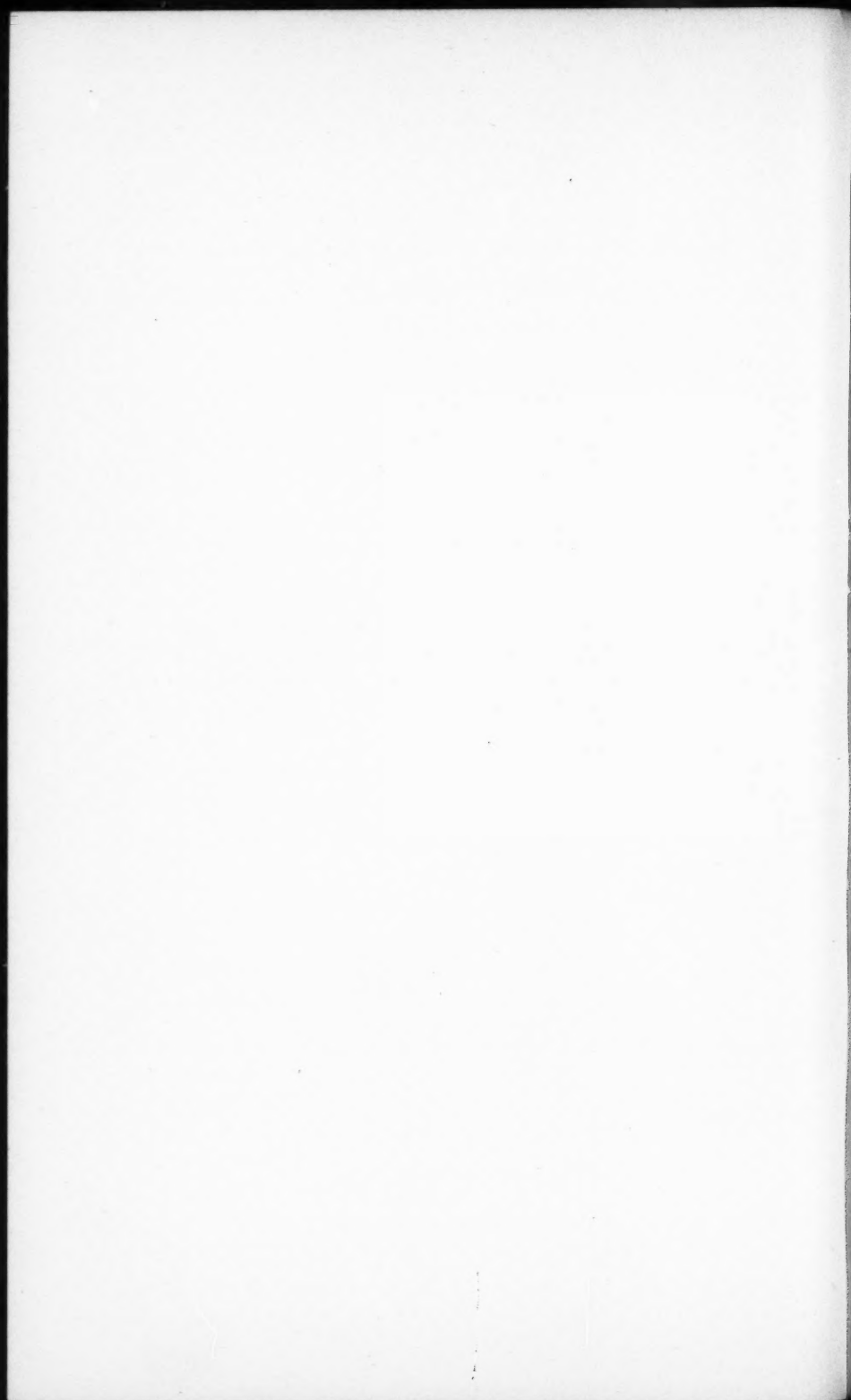
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CONTRIBUTIONS FROM THE JEFFERSON PHYSICAL LABORATORY,
HARVARD COLLEGE.

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SUMS OF THE CORRESPONDING MEMBERS OF
TWO PAIRS OF ORTHOGONAL FUNCTIONS OF
TWO VARIABLES ARE TO BE THEMSELVES
ORTHOGONAL.*

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If $\phi_1(x, y)$, $\phi_2(x, y)$ are the potential functions due to two columnar distributions of matter the lines of which are perpendicular to the xy plane, and if $\psi_1(x, y)$, $\psi_2(x, y)$ are conjugate to ϕ_1 and ϕ_2 , respectively, the families of curves obtained by equating ψ_1 and ψ_2 to parameters, are lines of force of the two distributions. Moreover, $\phi_1 + \phi_2$ is the potential function due to a combination of the two distributions, and the function $\psi_1 + \psi_2$ equated to a parameter gives the corresponding lines of force. The fact that if (ϕ_1, ψ_1) are any pair of conjugate functions and (ϕ_2, ψ_2) any other such pair, the functions $(a\phi_1 + b\phi_2, a\psi_1 + b\psi_2)$ are also conjugate — with similar facts for other classes of functions — lies at the foundation of the graphical methods so successfully used by Maxwell¹ and by others in drawing equipotential lines, and lines of force or flow, due to combinations of simple elements. If (ϕ_1, ψ_1) are merely a pair of orthogonal functions and (ϕ_2, ψ_2) another such pair, it is generally not true that $(\phi_1 + \phi_2, \psi_1 + \psi_2)$ are an orthogonal pair: thus (x, y) , $(x^2 + y^2, y/x)$ are pairs of orthogonal functions, but $x + x^2 + y^2$, $y + y/x$ are not orthogonal.

In certain classes of physical problems one encounters potential functions which are not themselves harmonic and the lines of which are not possible lines of any harmonic function, and it is often de-

¹ Maxwell, Treatise on Electricity and Magnetism, Vol. I, Ch. VII. Minchin, Uniplanar Kinematics, § 112. See also P. W. Bridgman, The electrostatic field surrounding two special columnar elements, These Proceedings, 41, 28.

sirable in cases where the analytical processes become too complex, to determine graphically the forms of lines of force or flow due to a combination of two simple elements. This note discusses briefly the conditions under which the ordinary method of procedure is possible.

Let (α, β) and (λ, μ) be two pairs of orthogonal functions of the two variables (x, y) , so that

$$\frac{\partial \alpha}{\partial x} \cdot \frac{\partial \beta}{\partial x} + \frac{\partial \alpha}{\partial y} \cdot \frac{\partial \beta}{\partial y} = 0, \quad (1)$$

$$\frac{\partial \lambda}{\partial x} \cdot \frac{\partial \mu}{\partial x} + \frac{\partial \lambda}{\partial y} \cdot \frac{\partial \mu}{\partial y} = 0; \quad (2)$$

then if $(\alpha + \lambda, \beta + \mu)$ are to form an orthogonal pair, the equation

$$\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \lambda}{\partial x} \right) \left(\frac{\partial \beta}{\partial x} + \frac{\partial \mu}{\partial x} \right) + \left(\frac{\partial \alpha}{\partial y} + \frac{\partial \lambda}{\partial y} \right) \left(\frac{\partial \beta}{\partial y} + \frac{\partial \mu}{\partial y} \right) = 0 \quad (3)$$

must be identically satisfied. Since (1) and (2) are true, (3) takes the form

$$\left(\frac{\partial \lambda}{\partial x} \cdot \frac{\partial \beta}{\partial x} + \frac{\partial \lambda}{\partial y} \cdot \frac{\partial \beta}{\partial y} \right) + \left(\frac{\partial \alpha}{\partial x} \cdot \frac{\partial \mu}{\partial x} + \frac{\partial \alpha}{\partial y} \cdot \frac{\partial \mu}{\partial y} \right) = 0. \quad (4)$$

If $h_\alpha, h_\beta, h_\lambda, h_\mu$ represent the values of the gradients of $\alpha, \beta, \lambda, \mu$, and if the angle at any point between the directions in which λ and β increase most rapidly be denoted by $[\lambda, \beta]$, (4) becomes

$$h_\lambda \cdot h_\beta \cdot \cos [\lambda, \beta] + h_\alpha \cdot h_\mu \cdot \cos [\alpha, \mu] = 0. \quad (5)$$

Whatever the sequence of the directions of the gradient vectors might be, the two angles which appear in (5) would be either equal or supplementary, and their cosines would be equal in absolute value, but the gradients themselves are intrinsically positive and the sequences must therefore be such that

$$h_\lambda / h_\mu = h_\alpha / h_\beta. \quad (6)$$

Suppose that in the case of two given pairs of orthogonal functions $(\alpha, \beta), (\lambda, \mu)$, the necessary condition (6) is satisfied, and that the

value of the gradient ratio, h_a/h_β , obtained from the given values of a and β , is the function Ω of x and y ; then

$$\left(\frac{\partial a}{\partial x}\right)^2 + \left(\frac{\partial a}{\partial y}\right)^2 = \Omega^2 \left(\frac{\partial \beta}{\partial x}\right)^2 + \Omega^2 \left(\frac{\partial \beta}{\partial y}\right)^2, \quad (7)$$

and if, for $(\partial a)^2/(\partial y)^2$ in this equation, we substitute the value

$$\left(\frac{\partial a}{\partial x}\right)^2 \left(\frac{\partial \beta}{\partial x}\right)^2 / \left(\frac{\partial \beta}{\partial y}\right)^2 \quad (8)$$

obtained from (1), it appears, since the gradient of a real function cannot vanish, that

$$\frac{\partial a}{\partial x} = \pm \Omega \left(\frac{\partial \beta}{\partial y}\right). \quad (9)$$

If for $(\partial a)^2/(\partial x)^2$ in (7), we substitute its equivalent

$$\left(\frac{\partial a}{\partial y}\right)^2 \left(\frac{\partial \beta}{\partial y}\right)^2 / \left(\frac{\partial \beta}{\partial x}\right)^2 \quad (10)$$

derived from (1), we shall learn that

$$\frac{\partial a}{\partial y} = \mp \Omega \left(\frac{\partial \beta}{\partial x}\right). \quad (11)$$

Either the upper signs or the lower signs must be used in (9) and (11).

If now we treat the equation

$$h_\lambda^2 = \Omega^2 h_\mu^2 \quad (12)$$

in a similar way we shall obtain the equations

$$\frac{\partial \lambda}{\partial x} = \pm \Omega \left(\frac{\partial \mu}{\partial y}\right), \quad (13)$$

$$\frac{\partial \lambda}{\partial y} = \mp \Omega \left(\frac{\partial \mu}{\partial x}\right), \quad (14)$$

and, so far as the relation (12) is concerned, we may use either the upper signs or the lower signs, but if (4) is to be identically satisfied, the *same* sign must be used in (9) and (13) and the sign opposite to this in (11) and (14). Equation (6), then, together with the proper choice of sequence of directions for the gradient vectors which corresponds to the convention with regard to signs just made, will ensure the orthogonality of $\alpha + \lambda$, $\beta + \mu$. For practical purposes, however, it is well to approach the problem from another side.

If (α, β) and (λ, μ) are given pairs of orthogonal functions, and if we denote the given scalar point functions obtained by dividing $\partial\beta/\partial x$ by $\partial\alpha/\partial y$, and by dividing $\partial\mu/\partial x$ by $\partial\lambda/\partial y$, by ζ and η , the equations (1) and (2) can be written in the forms

$$\frac{\partial\beta}{\partial x} \Big/ \frac{\partial\alpha}{\partial y} = - \frac{\partial\beta}{\partial y} \Big/ \frac{\partial\alpha}{\partial x} = \zeta, \quad (15)$$

and

$$\frac{\partial\mu}{\partial x} \Big/ \frac{\partial\lambda}{\partial y} = - \frac{\partial\mu}{\partial y} \Big/ \frac{\partial\lambda}{\partial x} = \eta, \quad (16)$$

or

$$\frac{\partial\beta}{\partial x} = \zeta \cdot \frac{\partial\alpha}{\partial y}, \quad \frac{\partial\beta}{\partial y} = -\zeta \cdot \frac{\partial\alpha}{\partial x}, \quad (17)$$

and

$$\frac{\partial\mu}{\partial x} = \eta \cdot \frac{\partial\lambda}{\partial y}, \quad \frac{\partial\mu}{\partial y} = -\eta \cdot \frac{\partial\lambda}{\partial x}. \quad (18)$$

If the values of the derivatives of β and μ given in (17) and (18) be substituted in (4) this equation becomes

$$(\zeta - \eta) \left(\frac{\partial\lambda}{\partial x} \cdot \frac{\partial\alpha}{\partial y} - \frac{\partial\alpha}{\partial x} \cdot \frac{\partial\lambda}{\partial y} \right) = 0, \quad (19)$$

and if $(\alpha + \lambda, \beta + \mu)$ are to be orthogonal, α and λ must be such as to satisfy it. If λ were expressible as a function of α , and μ as a function of β , the second factor would vanish, but this case is of no practical interest and (19) demands in general that ζ and η shall be identical, so that

$$\frac{\partial \beta}{\partial x} = \zeta \cdot \frac{\partial \alpha}{\partial y}, \quad \frac{\partial \beta}{\partial y} = -\zeta \cdot \frac{\partial \alpha}{\partial x}, \quad (17)$$

and

$$\frac{\partial \mu}{\partial x} = \zeta \cdot \frac{\partial \lambda}{\partial y}, \quad \frac{\partial \mu}{\partial y} - \zeta \cdot \frac{\partial \lambda}{\partial x} = 0. \quad (20)$$

If in these equations the arbitrary function ζ is made equal to unity, the conditions degenerate into the familiar definitions of any two pairs of conjugate functions.

In order that a single function (β) may exist the partial derivatives of which with respect to x and y shall be equal, respectively, to

$$\zeta \cdot \frac{\partial \alpha}{\partial y} \text{ and } -\zeta \cdot \frac{\partial \alpha}{\partial x},$$

it is necessary and it is sufficient that α and ζ should satisfy the condition

$$\frac{\partial}{\partial y} \left(\zeta \cdot \frac{\partial \alpha}{\partial y} \right) + \frac{\partial}{\partial x} \left(\zeta \cdot \frac{\partial \alpha}{\partial x} \right) = 0, \quad (21)$$

or

$$\frac{\partial \zeta}{\partial x} \cdot \frac{\partial \alpha}{\partial x} + \frac{\partial \zeta}{\partial y} \cdot \frac{\partial \alpha}{\partial y} + \zeta \cdot \nabla^2(\alpha) = 0. \quad (22)$$

In order that μ may exist, ζ and λ must satisfy the equation

$$\frac{\partial \zeta}{\partial x} \cdot \frac{\partial \lambda}{\partial x} + \frac{\partial \zeta}{\partial y} \cdot \frac{\partial \lambda}{\partial y} + \zeta \cdot \nabla^2(\lambda) = 0. \quad (23)$$

If $\log_e \zeta$ be represented by ϖ , the last two equations take the forms

$$\frac{\partial \varpi}{\partial x} \cdot \frac{\partial \alpha}{\partial x} + \frac{\partial \varpi}{\partial y} \cdot \frac{\partial \alpha}{\partial y} + \nabla^2(\alpha) = 0, \quad (24)$$

$$\frac{\partial \varpi}{\partial x} \cdot \frac{\partial \lambda}{\partial x} + \frac{\partial \varpi}{\partial y} \cdot \frac{\partial \lambda}{\partial y} + \nabla^2(\lambda) = 0, \quad (25)$$

and, if each of these be differentiated with respect to x and with respect to y , ϖ may be eliminated from the resulting equations and a necessary condition for α and λ obtained, which may be stated in the form of the determinantal equation —

$$\begin{vmatrix}
 \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} & 0 & 0 & 0 & \nabla^2(a) \\
 \frac{\partial \lambda}{\partial x} & \frac{\partial \lambda}{\partial y} & 0 & 0 & 0 & \nabla^2(\lambda) \\
 \frac{\partial^2 a}{\partial x^2} & \frac{\partial^2 a}{\partial x \cdot \partial y} & \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} & 0 & \frac{\partial}{\partial x} \left(\nabla^2(a) \right) \\
 \frac{\partial^2 \lambda}{\partial x^2} & \frac{\partial^2 \lambda}{\partial x \cdot \partial y} & \frac{\partial \lambda}{\partial x} & \frac{\partial \lambda}{\partial y} & 0 & \frac{\partial}{\partial x} \left(\nabla^2(\lambda) \right) \\
 \frac{\partial^2 a}{\partial x \cdot \partial y} & \frac{\partial^2 a}{\partial y^2} & 0 & \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} & \frac{\partial}{\partial y} \left(\nabla^2(a) \right) \\
 \frac{\partial^2 \lambda}{\partial x \cdot \partial y} & \frac{\partial^2 \lambda}{\partial y^2} & 0 & \frac{\partial \lambda}{\partial x} & \frac{\partial \lambda}{\partial y} & \frac{\partial}{\partial y} \left(\nabla^2(\lambda) \right)
 \end{vmatrix} = 0. \quad (26)$$

If a and λ happen to be harmonic, the elements of the last column vanish and the equation is satisfied, as it should be.

It is possible to factor the determinant, after it has been reduced, and if

$$\begin{aligned}
 L &\equiv \frac{\partial^2 a}{\partial x^2} \cdot \frac{\partial \lambda}{\partial x} - \frac{\partial^2 \lambda}{\partial x^2} \cdot \frac{\partial a}{\partial x} + \frac{\partial^2 a}{\partial x \cdot \partial y} \cdot \frac{\partial \lambda}{\partial y} - \frac{\partial^2 \lambda}{\partial x \cdot \partial y} \cdot \frac{\partial a}{\partial y}, \\
 M &\equiv \frac{\partial^2 a}{\partial x \cdot \partial y} \cdot \frac{\partial \lambda}{\partial x} - \frac{\partial^2 \lambda}{\partial x \cdot \partial y} \cdot \frac{\partial a}{\partial x} + \frac{\partial^2 a}{\partial y^2} \cdot \frac{\partial \lambda}{\partial y} - \frac{\partial^2 \lambda}{\partial y^2} \cdot \frac{\partial a}{\partial y}, \\
 N &\equiv \frac{\partial}{\partial x} \left(\nabla^2(a) \right) \cdot \frac{\partial \lambda}{\partial x} - \frac{\partial}{\partial x} \left(\nabla^2(\lambda) \right) \cdot \frac{\partial a}{\partial x} + \frac{\partial}{\partial y} \left(\nabla^2(a) \right) \cdot \frac{\partial \lambda}{\partial y} \\
 &\quad - \frac{\partial}{\partial y} \left(\nabla^2(\lambda) \right) \cdot \frac{\partial a}{\partial y},
 \end{aligned} \quad (27)$$

the condition of (26) demands either that a and λ satisfy the equation

$$\begin{vmatrix}
 \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y} & \nabla^2(a) \\
 \frac{\partial \lambda}{\partial x} & \frac{\partial \lambda}{\partial y} & \nabla^2(\lambda) \\
 L & M & N
 \end{vmatrix} = 0, \quad (28)$$

or else that α and λ have the same level curves: this last case, as being uninteresting, may be left out of account.

Sometimes (28) is more convenient than the unexpanded form of the same condition which follows immediately if we solve (24) and (25) for $\partial\varpi/\partial x$ and $\partial\varpi/\partial y$,

$$\frac{\partial\varpi}{\partial x} = \frac{\frac{\partial\alpha}{\partial y} \cdot \nabla^2(\lambda) - \frac{\partial\lambda}{\partial y} \cdot \nabla^2(\alpha)}{\frac{\partial\alpha}{\partial x} \cdot \frac{\partial\lambda}{\partial y} - \frac{\partial\alpha}{\partial y} \cdot \frac{\partial\lambda}{\partial x}}, \quad (29)$$

$$\frac{\partial\varpi}{\partial y} = \frac{\frac{\partial\lambda}{\partial x} \cdot \nabla^2(\alpha) - \frac{\partial\alpha}{\partial x} \cdot \nabla^2(\lambda)}{\frac{\partial\alpha}{\partial x} \cdot \frac{\partial\lambda}{\partial y} - \frac{\partial\alpha}{\partial y} \cdot \frac{\partial\lambda}{\partial x}}, \quad (30)$$

and equate the derivative with respect to y of the second member of the first equation to the derivative with respect to x of the second member of the other. This process yields the relation,

$$\frac{\partial}{\partial y} \left[\frac{\frac{\partial\alpha}{\partial y} \cdot \nabla^2(\lambda) - \frac{\partial\lambda}{\partial y} \cdot \nabla^2(\alpha)}{\frac{\partial\alpha}{\partial x} \cdot \frac{\partial\lambda}{\partial y} - \frac{\partial\alpha}{\partial y} \cdot \frac{\partial\lambda}{\partial x}} \right] = \frac{\partial}{\partial x} \left[\frac{\frac{\partial\lambda}{\partial x} \cdot \nabla^2(\alpha) - \frac{\partial\alpha}{\partial x} \cdot \nabla^2(\lambda)}{\frac{\partial\alpha}{\partial x} \cdot \frac{\partial\lambda}{\partial y} - \frac{\partial\alpha}{\partial y} \cdot \frac{\partial\lambda}{\partial x}} \right] \quad (31)$$

and it is possible to check the fact that (28) and (31) are equivalent by a straightforward but somewhat laborious comparison of the two.

If α and λ satisfy (31), a function ζ exists which satisfies (22) and (23), functions β and μ exist which satisfy (17) and (20), and (α, β) , (λ, μ) , $(\alpha + \lambda, \beta + \mu)$ are orthogonal pairs of functions.

If, for instance, both α and λ represent values in the xy plane of harmonic space functions (V, W) the level surfaces of which are surfaces of revolution about the x axis, so that

$$\frac{1}{y} \cdot \frac{\partial}{\partial y} \left(y \cdot \frac{\partial V}{\partial y} \right) + \frac{\partial^2 V}{\partial x^2} = 0 \quad (32)$$

with a similar equation for W ,

$$\nabla^2(\alpha) = -\frac{1}{y} \cdot \frac{\partial\alpha}{\partial y}, \quad \nabla^2(\lambda) = -\frac{1}{y} \cdot \frac{\partial\lambda}{\partial y}, \quad (33)$$

equation (31) is satisfied and

$$\frac{\partial \varpi}{\partial x} = 0, \quad \frac{\partial \varpi}{\partial y} = \frac{1}{y}, \quad \xi = cy. \quad (34)$$

In this case, if we put $c = 1$, β and μ are the Stokes functions corresponding to a and λ . If the level surfaces of the harmonic space functions, V and W , are surfaces of revolution about two different straight lines in the xy plane, the functions a and λ which represent the values of V and W in this plane do not in general satisfy (31).

Graphical superposition of the lines of force in the xy plane due to an infinitely long, homogeneous cylinder of revolution parallel to the axis, and to a homogeneous sphere with centre in the plane, will not in general yield the lines of force in the xy plane due to a combination of the two masses.

If a and λ are harmonic, any linear function (but no other than a linear function) of a is harmonic, and any two linear functions of a and λ satisfy (31). There generally exist, however, non-linear functions of a and λ which, although they are not harmonic, satisfy the condition. The functions $(x^2 - y^2)$, $(x^2 + y^2)^n$, the second of which is not harmonic, obey (31), as do the harmonic pair $(x^2 - y^2)$, $\log(x^2 + y^2)$.

As a simple example of the fact that a harmonic function and a function which is not even isothermal may satisfy the condition (31), we may consider $(2y^2 - x^2)$ and $(y^2 - x^2)$.

The non-isothermal functions $x^2 - ay^2$, $y^2 - ax^2$, which are solutions of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} - \frac{\partial V}{x \cdot \partial x} - \frac{\partial V}{y \cdot \partial y} = 0, \quad (35)$$

evidently satisfy the equation (31).

If a and λ are any two solutions of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + f'(x) \frac{\partial V}{\partial x} = 0, \quad (36)$$

where $f(x)$ is any given function of x , the condition (31) is satisfied and $\varpi = f(x)$.

If α and λ satisfy the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \alpha \frac{\partial V}{\partial x} + \lambda \frac{\partial V}{\partial y} = 0, \quad (37)$$

ϖ is of the form $\alpha x + \lambda y + c$.

In general α and λ must both be solutions of an equation of the form

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + P \cdot \frac{\partial V}{\partial x} + Q \cdot \frac{\partial V}{\partial y} = 0, \quad (38)$$

where P and Q are any functions of x and y such that $\partial P / \partial y = \partial Q / \partial x$.

The question whether if (α, β) and (λ, μ) are orthogonal pairs and $(\alpha + \lambda, \beta + \mu)$ is not an orthogonal pair, it is possible to find a function (B) of β , and a function (M) of μ such that $(\alpha + \lambda, B + M)$ shall form an orthogonal pair, has already been answered; for α and λ must satisfy (31) in any case, and if they do this, ϖ may be determined from (29) and (30) and B and M from (17) and (20).

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